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Lie symmetries of nonlinear multidimensional reaction–diffusion systems: I

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Abstract. A complete description of Lie symmetries is obtained for multidimensional semilinear systems of two reaction–diffusion equations. Moreover, a variety of Lie's ansätze and exact solutions of some particular reaction–diffusion systems, of a type that arises in mathematical biology for example, are constructed.

1. Introduction

In the present paper we shall consider nonlinear reaction–diffusion systems of the form

$$\begin{aligned} \lambda_1 U_t &= \Delta U + F(U, V) \\ \lambda_2 V_t &= \Delta V + G(U, V) \end{aligned} \qquad (1)$$

where F and G are arbitrary smooth functions, $U = U(t, x)$, $V = V(t, x)$ are unknown functions of $n + 1$ variables t, $x = (x_1, \ldots, x_n)$, Δ is the Laplacian and the subscript t on functions U and V denotes differentiation with respect to this variable.

The nonlinear system (1) generalizes many well known nonlinear second-order models and is used to describe various processes in physics [1], chemistry [2] and biology [3]. Nowadays systems of the form (1) are widely studied. There are many papers devoted to the investigation of existence and uniqueness problems, asymptotic behaviour of solutions and so on (see, e.g., [3, 4] and papers cited therein). On the other hand, to our knowledge there are only a few papers devoted to the search for Lie symmetries and exact solutions of systems of the form (1) (see [5–8]).

In a particular case, reaction–diffusion systems that are invariant with respect to the Galilei algebra and its extensions were described in [5, 8]. It was found that only systems of the form

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U f(\omega) \qquad \omega = U^{\lambda_2} V^{-\lambda_1} \\ \lambda_2 V_t &= \Delta V + V g(\omega) \end{aligned} \qquad (2)$$

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where f and g are arbitrary smooth functions on the variable ω, are invariant under the Galilei algebra $AG(1.n)$ with the following representation:

$$P_t = \frac{\partial}{\partial t} \qquad P_a = \frac{\partial}{\partial x_a} \qquad J_{ab} = x_a P_b - x_b P_a$$
$$Q_\lambda = \lambda_1 U \frac{\partial}{\partial U} + \lambda_2 V \frac{\partial}{\partial V} \qquad G_a = t P_a - \tfrac{1}{2} x_a Q_\lambda. \tag{3}$$

It should be stressed that although the technique of the Lie method is well known (see, e.g., [9–12]), it is a non-trivial problem to provide a complete description of the Lie symmetries of differential equations or systems containing *arbitrary functions*. For instance, Lie had calculated the maximal invariance algebra of the classical (1 + 1)-dimensional diffusion equation

$$U_t = U_{xx} \tag{4}$$

as far back as 1881 [13]. Nevertheless, the full classification of Lie symmetries for the single nonlinear reaction–diffusion equation

$$U_t = [A(U)U_x]_x + C(U) \tag{5}$$

was only calculated in 1982 [14], i.e. 100 years later! Note that in the recently published paper [15] the full classification of Lie symmetries of the nonlinear reaction–diffusion–convection equation

$$U_t = [A(U)U_x]_x + B(U)U_x + C(U) \tag{6}$$

where $A(U)$, $B(U)$ and $C(U)$ are arbitrary smooth functions, has been determined.

Having in mind a *complete description* of the Lie symmetry of system (1), we now summarize the main results of this paper.

In section 2, the classical Lie scheme is applied to find all possible Lie symmetries which the system (1) can admit. The main results of this section are presented in tables 1–5. Note that we present the full classification only for the case $\lambda_1 \neq \lambda_2$ with λ_1, for example, non-vanishing. It turns out that the case $\lambda_1 = \lambda_2$ is very special and we are going to devote to this case the second part of this work.

In section 3, the (1 + 1)-dimensional reaction–diffusion system, preserving the Lie symmetry of the linear diffusion equation, namely

$$\lambda_1 U_t = U_{xx} + \beta_1 U (U^{-\lambda_2} V^{\lambda_1})^{4/(\lambda_1-\lambda_2)}$$
$$\lambda_2 V_t = V_{xx} + \beta_2 V (U^{-\lambda_2} V^{\lambda_1})^{4/(\lambda_1-\lambda_2)} \tag{7}$$

is considered in detail. All non-equivalent Lie ansätze are presented, together with formulae for the multiplication of solutions and examples of exact solutions.

2. Lie symmetries of system (1)

It is easily checked that the system (1) is invariant under the operators P_a, J_{ab} and P_t (see (3)) for arbitrary functions F and G. The operators P_a and J_{ab} form the well known Euclid algebra $AE(n)$. Its extension by the operator P_t we will denote as the $AE(1.n)$ algebra. Following [15], this algebra is called the *trivial Lie algebra* of the system (1). Thus, we aim to find all pairs of functions (F, G) that lead to extensions of the trivial Lie algebra for systems of the form (1). Note that we consider only *nonlinear* systems, particularly because linear equations are amenable to numerous classical methods (the Fourier method, the method of Laplace transformation and so on).

Now let us formulate a theorem which gives complete information on the classical symmetry of the system (1).

Theorem 1. *All possible maximal algebras of invariance (MAI) of the system (1) for any fixed pair of functions F, G and $\lambda_1 \neq \lambda_2$ are presented in tables 1–5. Any other system of the form (1) with non-trivial Lie symmetry is reduced by a local substitution to one of those given in tables.*

Proof of theorem 1. This proof is based on the classical Lie scheme (see, e.g., [9–12]) and is very cumbersome because the system (1) contains two arbitrary functions of two variables. Here we give only an outline of how the proof proceeds. According to the Lie approach, the system (1) is considered as a manifold (S_1, S_2)

$$\begin{aligned} S_1 &\equiv \lambda_1 U_t - \Delta U - F(U, V) = 0 \\ S_2 &\equiv \lambda_2 V_t - \Delta V - G(U, V) = 0 \end{aligned} \tag{8}$$

in the space of the following variables:

$$t, x, U, V, U_t, V_t, U_1, \ldots, U_n, V_1, \ldots, V_n, U_{11}, \ldots, U_{nn}, V_{11}, \ldots, V_{nn}$$

where subscripts $1, \ldots, n$ to the functions U and V denote differentiation with respect to the variables $x_1, \ldots, x_n$.

System (1) is invariant under the transformations generated by the infinitesimal operator

$$X = \xi^0(t, x, U, V)\partial_t + \xi^a(t, x, U, V)\partial_{x_a} + \eta^U(t, x, U, V)\partial_U + \eta^V(t, x, U, V)\partial_V \tag{9}$$

when the following invariance conditions are satisfied:

$$\begin{aligned} \underset{11}{X} S_1 &\equiv \underset{11}{X}(\lambda_1 U_t - \Delta U - F(U, V))\Big|_{\substack{S_1=0\\S_2=0}} = 0 \\ \underset{11}{X} S_2 &\equiv \underset{11}{X}(\lambda_2 V_t - \Delta V - G(U, V))\Big|_{\substack{S_1=0\\S_2=0}} = 0. \end{aligned} \tag{10}$$

The operator $\underset{11}{X}$ is the second prolongation of the operator X, i.e.

$$\underset{11}{X} = X + \rho_t^U \frac{\partial}{\partial U_t} + \rho_t^V \frac{\partial}{\partial V_t} + \rho_a^U \frac{\partial}{\partial U_a} + \rho_a^V \frac{\partial}{\partial V_a} + \sigma_{ab}^U \frac{\partial}{\partial U_{ab}} + \sigma_{ab}^V \frac{\partial}{\partial V_{ab}} \tag{11}$$

where the coefficients ρ and σ with relevant subscripts are calculated by well known formulae (see, e.g., [12]) and summation is assumed from 1 to n over the repeated indices a, b. Substituting (11) into (10), we can split this relation into separate parts for the derivatives

$$U_t, V_t, U_1, \ldots, U_n, V_1, \ldots, V_n, U_{11}, \ldots, U_{nn}, V_{11}, \ldots, V_{nn}.$$

Finally, after the relevant calculations, we obtain the following system for the coefficients

$\xi^0, \xi^a, \eta^U, \eta^V$ of the operator X:

$$\begin{aligned}
&\frac{\partial\xi^0}{\partial x_a} = \frac{\partial\xi^0}{\partial U} = \frac{\partial\xi^0}{\partial V} = 0 \qquad a = 1,\dots,n \\
&\frac{\partial\xi^a}{\partial U} = \frac{\partial\xi^a}{\partial V} = 0 \qquad \frac{\partial\xi^b}{\partial x_a} + \frac{\partial\xi^a}{\partial x_b} = 0 \qquad a,b = 1,\dots,n \quad a \neq b \\
&\frac{\partial\xi^0}{\partial t} = 2\frac{\partial\xi^a}{\partial x_a} \qquad a = 1,\dots,n \\
&2\frac{\partial^2\eta^U}{\partial x_a \partial U} = \Delta\xi^a - \lambda_1\frac{\partial\xi^a}{\partial t} \qquad a = 1,\dots,n \\
&2\frac{\partial^2\eta^V}{\partial x_a \partial V} = \Delta\xi^a - \lambda_2\frac{\partial\xi^a}{\partial t} \qquad a = 1,\dots,n \\
&\eta^U = E^1(t,x)U + q^1(t)V + P^1(t,x) \\
&\eta^V = E^2(t,x)V + q^2(t)U + P^2(t,x);
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\lambda_1\frac{\partial\eta^U}{\partial V} = \lambda_2\frac{\partial\eta^U}{\partial V} \\
&\lambda_1\frac{\partial\eta^V}{\partial U} = \lambda_2\frac{\partial\eta^V}{\partial U};
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\lambda_1\frac{\partial\eta^U}{\partial t} - \Delta\eta^U + F\left(\frac{\partial\eta^U}{\partial U} - \frac{\partial\xi^0}{\partial t}\right) + G\frac{\partial\eta^U}{\partial V} = \eta^U\frac{\partial F}{\partial U} + \eta^V\frac{\partial F}{\partial V} \\
&\lambda_2\frac{\partial\eta^V}{\partial t} - \Delta\eta^V + G\left(\frac{\partial\eta^V}{\partial V} - \frac{\partial\xi^0}{\partial t}\right) + F\frac{\partial\eta^V}{\partial U} = \eta^U\frac{\partial G}{\partial U} + \eta^V\frac{\partial G}{\partial V}
\end{aligned} \tag{14}$$

where $E^k(t,x)$, $q^k(t)$, $P^k(t,x)$, $k = 1, 2$ are arbitrary smooth functions.

One can see that the subsystem (12) is an overdetermined one and it is possible to construct its general solution, namely:

$$\begin{aligned}
&\xi^0 = 2A(t) \\
&\xi^a = c_{ab}x_b + \dot{A}(t)x_a + g_a(t) \qquad a,b = 1,\dots,n \quad a \neq b \\
&\eta^U = -\tfrac{1}{2}\lambda_1\left(\tfrac{1}{2}|x|^2\ddot{A}(t) + \dot{g}_a(t)x_a\right)U + r^1(t)U + q^1(t)V + P^1(t,x) \\
&\eta^V = -\tfrac{1}{2}\lambda_2\left(\tfrac{1}{2}|x|^2\ddot{A}(t) + \dot{g}_a(t)x_a\right)V + r^2(t)V + q^2(t)U + P^2(t,x)
\end{aligned} \tag{15}$$

where $A(t), g_a(t), a = 1,\dots,n, r^k(t), q^k(t), P^k(t,x), k = 1, 2$ are arbitrary smooth functions, $c_{ab}+c_{ba} = 0$, $c_{ab} \in \mathbb{R}$, and the dots over the functions denote differentiation with respect to the variable t. Taking into account (15), we can consider equations (13) and (14) as classification equations to find the pairs of (F, G) for which the system (1) has a non-trivial Lie symmetry.

It can be seen that there are three main cases which lead to essentially different types of Lie symmetry of the system (1), namely:

(a) $\lambda_1 \neq \lambda_2, \lambda_1\lambda_2 \neq 0$;
(b) $\lambda_1\lambda_2 = 0$;
(c) $\lambda_1 = \lambda_2$.

In the first case, it follows from (13) that $q^1 = q^2 = 0$ and then the subsystem (14) is non-coupled. Moreover, both equations of this subsystem have the same structure. In the second case, it follows again that q^1 and q^2 vanish. Without losing generality, we can assume $\lambda_1 = \lambda \neq 0$, $\lambda_2 = 0$ and then the second equation of (14) is simpler in structure than the first one. The case $\lambda_1 = \lambda_2 = 0$ is not considered here because then the evolution system degenerates into an elliptic system. In the third case, equations (13) are satisfied by arbitrary functions q^1 and q^2, so that the subsystem (14) is coupled. This case will be considered in a subsequent paper.

Case (a). This is the most general and interesting case. Taking into account (15), one sees that the most non-trivial symmetry can occur when

$$E \equiv \tfrac{1}{2}|x|^2\ddot{A}(t) + \dot{g}_a(t)x_a \neq 0. \tag{16}$$

Substituting coefficients (15) into (14) and solving the system obtained using the restriction (16), we find all possible extensions of the trivial Lie algebra, listed in table 1. Note that we have shown only local non-equivalent systems. The corresponding local substitutions have the form

$$\begin{aligned} U &\to c_1 \exp(c_3 t)U + c_{10} \\ V &\to c_2 \exp(c_4 t)V + c_{20} \end{aligned} \tag{17}$$

where the coefficients c with subscripts are determined by the form of the system in question.

If the restriction (16) is not valid, i.e. $E = 0$, then (15) takes the form

$$\begin{aligned} \xi^0 &= 2A_1 t + d_0 \\ \xi^a &= c_{ab}x_b + A_1 x_a + d_a \qquad a, b = 1, \ldots, n \quad a \neq b \\ \eta^U &= r^1(t)U + P^1(t, x) \\ \eta^V &= r^2(t)V + P^2(t, x) \end{aligned} \tag{18}$$

where $A_1, d_0, d_1, \ldots, d_n$ are arbitrary parameters. Again one notes that the widest symmetry occurs in the case $A_1 \neq 0$. In this case all systems of the form (1) that are invariant under scaling transformations with respect to the independent variables (of the form $t' = \varepsilon^2 t$, $x'_a = \varepsilon x_a$, $a = 1, \ldots, n$, $\varepsilon \in \mathbb{R}$) can be described. These results are summarized in table 3. Note that two additional cases (numbers 6 and 12), that belong only to case (b), are also listed in table 3. The set of local substitutions that reduce any system of the form (1) with the above-mentioned symmetry to one of the cases of table 3 has the form

$$\begin{aligned} U &\to c_1 \exp(c_3 t)U + c_5|x|^2 + c_7 t + c_{10} \\ V &\to c_2 \exp(c_4 t)V + c_6|x|^2 + c_8 t + c_{20}. \end{aligned} \tag{19}$$

Finally, if $A_1 = 0$ then the trivial Lie algebra $AE(1.n)$ of the system (1) can be extended only by operators of the form

$$\begin{aligned} X_1^\infty &= P^1(t, x)\partial_U \qquad & X_2^\infty &= P^2(t, x)\partial_V \\ I_1^\infty &= T^1(t)U\partial_U \qquad & I_2^\infty &= T^2(t)V\partial_V \end{aligned} \tag{20}$$

where $T^k(t)$, $P^k(t, x)$, $k = 1, 2$ are some functions or constants, and by their linear combinations. All possible functions have been found and the results are summarized in tables 4 and 5. Again, additional cases (see numbers 3–6 and 12, 13 in table 4 and numbers 2, 4 and 5 in table 5) that belong only to case (b) are also listed in tables 4 and 5. The set of local substitutions that reduce any system of the form (1) with symmetry (20) to one of the cases of tables 4 and 5 again has the form (19).

Table 1. Galilei-invariant and pseudo-Galilean-invariant systems of the form (1) at $\lambda_k \neq 0, k = 1, 2$.

	Systems	Restrictions	Basic operators of MAI
1	$\lambda_1 U_t = \Delta U + U f(\omega)$ $\lambda_2 V_t = \Delta V + V g(\omega)$	$\omega = U^{-\lambda_2} V^{\lambda_1}$	$AE(1.n)$, $Q_\lambda = \lambda_1 U \partial_U + \lambda_2 V \partial_V$ $G_a = t P_a - \frac{1}{2} x_a Q_\lambda$
2	$\lambda_1 U_t = \Delta U + \beta_1 U \omega^\alpha$ $\lambda_2 V_t = \Delta V + \beta_2 V \omega^\alpha$	$\alpha \neq 0$ $\beta_1 \neq 0$	$AE(1.n)$, Q_λ, G_a $D = 2t P_t + x_a P_a - \dfrac{2}{\lambda_1 \alpha} V \partial_V$
3	$\lambda_1 U_t = \Delta U + \beta_1 U [U^{-\lambda_2} V^{\lambda_1}]^{\gamma_n}$ $\lambda_2 V_t = \Delta V + \beta_2 V [U^{-\lambda_2} V^{\lambda_1}]^{\gamma_n}$	$\gamma_n = \dfrac{4}{n(\lambda_1 - \lambda_2)}$ $\beta_1 \neq 0$	$AE(1.n)$, Q_λ, G_a $D = 2t P_t + x_a P_a - I_n$ $\Pi = t^2 P_t + t x_a P_a - \frac{1}{4} \lvert x \rvert^2 Q_\lambda - t I_n$
4	$\lambda_1 U_t = \Delta U + U(\beta_1 + \lambda_1^2 \beta_0 \log \omega)$ $\lambda_2 V_t = \Delta V + V(\beta_2 + \lambda_2^2 \beta_0 \log \omega)$	$\beta_0 \neq 0$	$AE(1.n)$, Q_λ, G_a $Y = \dfrac{2}{n} I_n + \beta_0(\lambda_1 - \lambda_2) t Q_\lambda$
5	$\lambda_1 U_t = \Delta U + U(\beta_1 + \beta_{10} \log \omega)$ $\lambda_2 V_t = \Delta V + V(\beta_2 + \beta_{20} \log \omega)$	$\beta_{20} \neq 0$ $\beta_{10} \lambda_2^2 \neq \beta_{20} \lambda_1^2$	$AE(1.n)$, Q_λ, G_a $\mathcal{Q}_\beta = Q_\beta \exp\left(\dfrac{\beta_{20}\lambda_1^2 - \beta_{10}\lambda_2^2}{\lambda_1 \lambda_2} t\right)$
6	$\lambda_1 U_t = \Delta U + U(\lambda_1 \beta \log U + f(\omega))$ $\lambda_2 V_t = \Delta V + V(\lambda_2 \beta \log V + g(\omega))$	$\beta \neq 0$ $\omega = U^{-\lambda_2} V^{\lambda_1}$	$AE(1.n)$, $\mathcal{Q}_\lambda = \exp(\beta t) Q_\lambda$ $\mathcal{G}_a = \exp(\beta t) P_a - \frac{1}{2} \beta x_a \mathcal{Q}_\lambda$
7	$\lambda_1 U_t = \Delta U$ $+U(\beta_1 + \lambda_1 \beta \log U + \beta_{10} \log \omega)$ $\lambda_2 V_t = \Delta V$ $+V(\beta_2 + \lambda_2 \beta \log V + \beta_{20} \log \omega)$	$\beta \neq 0$ $\beta \lambda_1 \lambda_2 = \beta_{10} \lambda_2^2 - \beta_{20} \lambda_1^2$	$AE(1.n)$, $\mathcal{Q}_\lambda$, $\mathcal{G}_a$ $Q_\beta = \beta_{10} \lambda_2 U \partial_U + \beta_{20} \lambda_1 V \partial_V$
8	$\lambda_1 U_t = \Delta U$ $+U(\beta_1 + \lambda_1 \beta \log U + \beta_{10} \log \omega)$ $\lambda_2 V_t = \Delta V$ $+V(\beta_2 + \lambda_2 \beta \log V + \beta_{20} \log \omega)$	$\beta \neq 0$ $\beta_{10} \lambda_2^2 \neq \beta_{20} \lambda_1^2$ $\beta_{10} \lambda_2^2 - \beta_{20} \lambda_1^2 \neq \beta \lambda_1 \lambda_2$	$AE(1.n)$, $\mathcal{Q}_\lambda$, $\mathcal{G}_a$ $\mathcal{Q}_\beta^1 = \exp(\beta t) \mathcal{Q}_\beta$
9	$\lambda_1 U_t = \Delta U$ $+U(\beta_1 + \lambda_1 \beta \log U + \lambda_1^2 \beta_0 \log \omega)$ $\lambda_2 V_t = \Delta V$ $+V(\beta_2 + \lambda_2 \beta \log V + \lambda_2^2 \beta_0 \log \omega)$	$\beta \beta_0 \neq 0$	$AE(1.n)$, $\mathcal{Q}_\lambda$, $\mathcal{G}_a$ $\mathcal{Y} = \exp(\beta t) \dfrac{2}{n} I_n + \beta_0(\lambda_1 - \lambda_2) t \mathcal{Q}_\lambda$

Remark 1. In table 1, the following designation is introduced: $I_n = \frac{1}{2} n (U \partial_U + V \partial_V)$.

All possible systems that have non-trivial Lie symmetry have now been described in case (a).

Case (b). This case is more cumbersome than case (a). Nevertheless, it is possible to make use of the results of the investigation of case (a). It turns out that for $E = 0$ most of the pairs of nonlinearities (F, G) that lead to non-trivial Lie symmetries can be obtained from the relevant cases of tables 3–5, if there one formally puts $\lambda_2 = 0$. On the other hand, cases 6 and 12 in table 3, 3–6 and 12–13 in table 4, and 2, 4 and 5 in table 5 arise only when $\lambda_2 = 0$.

When the restriction (16) holds, the full classification gives a set of new nonlinearities (F, G) which lead to non-trivial Lie symmetries. We have found 16 relevant cases, which are listed in table 2. Note that only some of them can be obtained formally from the corresponding cases in table 1.

The set of local substitutions that reduce any system of the form (1), with Lie operator(s) satisfying (16), to one of the cases of table 2 has the form

$$\begin{aligned} U &\to c_1 \exp(c_3 t) U + c_{10} \\ V &\to c_2 V + c_4 t + c_5 t |x|^2 + c_{20}. \end{aligned} \tag{21}$$

The sketch of the proof is now completed. □

Remark 2. In tables 1–5, $f(\omega)$, $g(\omega)$ and $T(t)$ are arbitrary smooth functions, while $P_1(t,x)$, $P_2(t,x)$, $P_{\beta_1}(t,x)$, $P_{\beta_2}(t,x)$, $P_0(t,x)$, $R(x)$ and $R_0(x)$ are arbitrary solutions of the linear equations

$$\begin{aligned} \lambda_1 P_t &= \Delta P \\ \lambda_2 P_t &= \Delta P \\ \lambda_1 P_t &= \Delta P + \beta_1 P \\ 0 &= \Delta P + \beta_2 P \\ 0 &= \Delta P \\ \Delta R &= \frac{\lambda_2\beta_1 - \lambda_1\beta_2}{\lambda_1 - \lambda_2} R \\ \Delta R_0 &= 0 \end{aligned} \tag{22}$$

respectively.

Remark 3. A number of the cases that are noted as special with respect to their symmetry properties also arise in applications. Power-law nonlinearities (such as those arising in table 1, cases 2 and 3 and table 2, cases 2 and 3) are frequently adopted in chemical reaction modelling, for example, while combinations of exponentials and power laws (akin to those in table 2, cases 5 and 9 and table 3, case 9) arise when the effects of variations in temperature on reaction rates are accounted for, such as in the modelling of combustion processes.

It is worth commenting on systems and Lie algebras listed in table 1. It turns out that, in contrast to the scalar case, there are many Galilei-invariant systems of the form (1) (see case 1 in table 1). Cases 2 and 3 of table 1 are natural continuations of case 1, because the extended Galilei algebra $AG_1(1.n)$ and the generalized Galilei algebra $AG_2(1.n)$ are known to be standard extensions of the classical Galilei algebra $AG(1.n)$ (for details see [12, 16]). Moreover, the system (1) is invariant with respect to the $AG_2(1.n)$ algebra only in the case of a particular power nonlinearity that depends on the values of λ_1, λ_2 and n. Cases 4 and 5 represent two new extensions of the Galilean algebra by the operators Y and Q_β, respectively. Both extensions are different from the $AG_1(1.n)$ algebra because they contain the commutative relations $[P_t, Y] = \beta_0(\lambda_1 - \lambda_2) Q_\lambda$ and $[P_t, Q_\beta] = (\beta_{20}\lambda_1^2 - \beta_{10}\lambda_2^2) Q_\beta / \lambda_1\lambda_2$, respectively.

In contrast to cases 1–5, case 6 has a direct analogue among single nonlinear reaction–diffusion equations. Indeed, according to [14],

$$U_t = \Delta U + \beta_0 U \log U \qquad \beta_0 \neq 0$$

Table 2. Galilei-invariant and pseudo-Galilean-invariant systems of the form (1) at $\lambda \equiv \lambda_1 \neq 0, \lambda_2 = 0$.

	Systems	Restrictions	Basic operators of MAI
1	$\lambda U_t = \Delta U + Uf(V)$ $0 = \Delta V + g(V)$		$AE(1.n)$, $Q = U\partial_U$ $G_a = tP_a - \frac{1}{2}\lambda x_a Q$
2	$\lambda U_t = \Delta U$ $0 = \Delta V + g(V)$		$AE(1.n)$, Q, G_a $X_1^\infty = P_1(t,x)\partial_U$
3	$\lambda U_t = \Delta U$ $0 = \Delta V + \beta_2 V^{1+\gamma}$	$\beta_2\gamma \neq 0$ $\gamma \neq -1$	$AE(1.n)$, Q, G_a, X_1^∞ $D_1 = 2tP_t + x_aP_a - \frac{2}{\gamma}VP_V$ $\Pi_1 = t^2P_t + tx_aP_a$ $-(\frac{1}{4}\lambda\|x\|^2 + \frac{1}{2}nt)U\partial_U - \frac{2}{\gamma}tV\partial_V$
4	$\lambda U_t = \Delta U + \beta_1 UV^\gamma$ $0 = \Delta V + \beta_2 V^{1+\gamma}$	$\gamma\beta_1\beta_2 \neq 0$	$AE(1.n)$, Q, G_a D_1, Π_1
5	$\lambda U_t = \Delta U + \beta_1 U\exp(\gamma V)$ $0 = \Delta V + \beta_2\exp(\gamma V)$	$\gamma\beta_1 \neq 0$	$AE(1.n)$, Q, G_a $D_2 = 2tP_t + x_aP_a - \frac{2}{\gamma}P_V$ $\Pi_2 = t^2P_t + tx_aP_a$ $-(\frac{1}{4}\lambda\|x\|^2 + \frac{1}{2}nt)U\partial_U - \frac{2}{\gamma}t\partial V$
6	$\lambda U_t = \Delta U$ $0 = \Delta V + \beta_2\exp(\gamma V)$	$\beta_2\gamma \neq 0$	$AE(1.n)$, Q, G_a D_2, Π_2, X_1^∞
7	$\lambda U_t = \Delta U + \gamma U\log V$ $0 = \Delta V + \beta_2 V$	$\gamma \neq 0$	$AE(1.n)$, Q, G_a $Y^\infty = \left(\gamma\int T(t)\,\mathrm{d}t\right)U\partial_U + \lambda T(t)V\partial_V$
8	$\lambda U_t = \Delta U + Uf(\omega)$ $0 = \Delta V + g(\omega)$	$\alpha \neq 0$ $\omega = U^\alpha\exp V$	$AE(1.n)$, $Q_\alpha = U\partial_U - \alpha\partial_V$ $G_a^\alpha = tP_a - \frac{1}{2}\lambda x_a Q_\alpha$
9	$\lambda U_t = \Delta U + \beta_1 U^{\alpha+1}\exp V$ $0 = \Delta V + \beta_2 U^\alpha\exp V$	$\alpha\beta_2 \neq 0$	$AE(1.n)$, Q_α, G_a^α $D_0 = 2tP_t + x_aP_a - 2\partial_V$
10	$\lambda U_t = \Delta U$ $0 = \Delta V + \alpha_2\log U + \beta_2 V$	$\alpha_2\beta_2 \neq 0$	$AE(1.n)$, Q_α, G_a^α at $\alpha = \alpha_2/\beta_2$ $X_{\beta_2}^\infty = P_{\beta_2}(t,x)\partial_V$
11	$\lambda U_t = \Delta U$ $0 = \Delta V + \alpha_2\log U$	$\alpha_2 \neq 0$	$AE(1.n)$, $X_0^\infty = P_0(t,x)\partial_V$ $Q_{esp} = U\partial_U - \frac{\alpha_2}{2n}\|x\|^2\partial_V$ $G_a^{esp} = tP_a - \frac{1}{2}\lambda x_a\left(Q_{esp} + \frac{\alpha_2}{3n}x_a^2\partial_V\right)$ $D = 2tP_t + x_aP_a + 2V\partial_V$ $\Pi_{esp} = tD - t^2P_t - (\frac{1}{4}\lambda\|x\|^2 + \frac{1}{2}nt)U\partial_U$ $+\left(\frac{\lambda\alpha_2}{16(n+2)}(\|x\|^2)^2 + \frac{\alpha_2 t}{4}\|x\|^2\right)\partial_V$
12	$\lambda U_t = \Delta U + \gamma U\log U + Uf(V)$ $0 = \Delta V + g(V)$	$\gamma \neq 0$	$AE(1.n)$, $\mathcal{Q} = \exp\left(\frac{\gamma}{\lambda}t\right)U\partial_U$ $\mathcal{G}_a = \exp\left(\frac{\gamma}{\lambda}t\right)P_a - \frac{1}{2}\gamma x_a\mathcal{Q}$
13	$\lambda U_t = \Delta U + \gamma U\log U$ $0 = \Delta V + \beta_2 V$	$\gamma \neq 0$	$AE(1.n)$, $\mathcal{Q}$, $\mathcal{G}_a$ $I^\infty = T(t)V\partial_V$, $X_{\beta_2}^\infty = P_{\beta_2}(t,x)\partial_V$
14	$\lambda U_t = \Delta U + U(\gamma\log U + \beta_{10}\log V)$ $0 = \Delta V + \beta_2 V$	$\gamma\beta_{10} \neq 0$	$AE(1.n)$, $\mathcal{Q}$, $\mathcal{G}_a$ $\mathcal{Y}^\infty = \exp\left(\frac{\beta_{10}}{\lambda}t\right)Y^\infty$

Table 2. Continued.

	Systems	Restrictions	Basic operators of MAI
15	$\lambda U_t = \Delta U + U(\gamma \log U + f(\omega))$	$\gamma\alpha \neq 0$	$AE(1.n)$, $\mathcal{Q}^\alpha = \exp\left(\frac{\gamma}{\lambda}t\right)(U\partial_U - \alpha\partial_V)$
	$0 = \Delta V + g(\omega)$	$\omega = U^\alpha \exp V$	$\mathcal{G}_a^\alpha = \exp\left(\frac{\gamma}{\lambda}t\right)P_a - \frac{1}{2}\gamma x_a \mathcal{Q}^\alpha$
16	$\lambda U_t = \Delta U + \gamma U \log U$	$\alpha_2\beta_2\gamma \neq 0$	$AE(1.n)$, $X^\infty_{\beta_2}$
	$0 = \Delta V + \alpha_2 \log U + \beta_2 V$		$\mathcal{Q}^\alpha, \mathcal{G}_a^\alpha$ at $\alpha = \dfrac{\alpha_2}{\beta_2}$

is invariant with respect to the $A\mathcal{G}(1.n)$ algebra with basic operators

$$
\begin{aligned}
&P_t = \frac{\partial}{\partial t} \qquad P_a = \frac{\partial}{\partial x_a} \qquad J_{ab} = x_a P_b - x_b P_a \\
&\mathcal{Q} = \exp(\beta_0 t)U\partial_U \qquad \mathcal{G}_a = \exp(\beta_0 t)P_a - \tfrac{1}{2}\beta_0 x_a \mathcal{Q}.
\end{aligned}
\tag{23}
$$

Here we call this algebra the pseudo-Galilean algebra. Note that the $A\mathcal{G}(1.n)$ algebra is different from the Galilei algebra because it contains the commutative relations $[P_t, \mathcal{Q}] = \beta_0\mathcal{Q}$ and $[P_t, \mathcal{G}_a] = \beta_0\mathcal{G}_a$. Finally, cases 7–9 of table 1 are natural continuations of case 6, because they represent three new extensions of the pseudo-Galilean algebra.

3. Lie ansätze and solutions of the nonlinear reaction–diffusion system (7)

Consider the nonlinear reaction–diffusion system (see table 1, case 3)

$$
\begin{aligned}
\lambda_1 U_t &= \Delta U + \beta_1 U^{1-\lambda_2\gamma_n} V^{\lambda_1\gamma_n} \\
\lambda_2 V_t &= \Delta V + \beta_2 V^{1+\lambda_1\gamma_n} U^{-\lambda_2\gamma_n}
\end{aligned}
\tag{24}
$$

where $\gamma_n = 4/(n(\lambda_1 - \lambda_2))$, $\lambda_1 \neq \lambda_2$. It preserves the $AG_2(1.n)$ symmetry of the linear diffusion system

$$
\begin{aligned}
\lambda_1 U_t &= \Delta U \\
\lambda_2 V_t &= \Delta V.
\end{aligned}
\tag{25}
$$

It should be stressed that this is a non-trivial result since there is no scalar nonlinear generalization (5) of the linear equation (4) which preserves its $AG_2(1.1)$ symmetry.

In this section we shall deal with the $(1 + 1)$-dimensional version of the system (24), i.e. (7). This system with $\lambda_1 - \lambda_2 < 0$ can be considered as a limiting case of a model used to describe a biological pattern arising in *hydra* [17, 18], namely

$$
\begin{aligned}
\lambda_1 U_t &= U_{xx} + \beta_1 U(U^{\alpha_2} V^{-\alpha_1} - d_1) \\
\lambda_2 V_t &= V_{xx} + \beta_2 V(U^{\alpha_4} V^{-\alpha_3} - d_2)
\end{aligned}
\tag{26}
$$

where the coefficients are some non-negative parameters. Note that in the case $\alpha_2/\alpha_1 = \alpha_4/\alpha_3 = \lambda_2/\lambda_1$, the system (26) is invariant with respect to the Galilei algebra $AG(1.n)$ (see table 1, case 1). In the case of the additional restrictions $d_1 = d_2 = 0, \alpha_2 = \alpha_4, \alpha_1 = \alpha_3$, it has the $AG_1(1.n)$ symmetry (see table 1, case 2).

Thus, it seems reasonable to construct Lie ansätze and to seek exact solutions of the nonlinear system (7). With this in mind, consider its Lie symmetry generated by the basic

Table 3. Scale-invariant systems of the form (1).

	Systems	Restrictions	Basic operators of MAI
1	$\lambda_1 U_t = \Delta U + U^{1-\alpha_0} f(\omega)$ $\lambda_2 V_t = \Delta V + V U^{-\alpha_0} g(\omega)$	$\alpha_0 \neq 0$ $\omega = U^{-\gamma} V$	$AE(1.n)$, $D_1 = 2t\partial_t + x_a\partial_a$ $+\dfrac{2}{\alpha_0}(U\partial_U + \gamma V\partial_V)$
2	$\lambda_1 U_t = \Delta U + \beta_1 U^{1-\alpha_0}$ $\lambda_2 V_t = \Delta V + \beta_2 U^{\gamma-\alpha_0}$	$\alpha_0\beta_1\beta_2 \neq 0$	$AE(1.n)$, D_1 $X_2^\infty = P_2(t, x)\partial_V$
3	$\lambda_1 U_t = \Delta U + \beta_1 U^{1-\alpha_0} V^{\alpha_1}$ $\lambda_2 V_t = \Delta V + \beta_2 U^{-\alpha_0} V^{1+\alpha_1}$	$\alpha_0\alpha_1\beta_1 \neq 0$ $\alpha_0\lambda_1 \neq \alpha_1\lambda_2$	$AE(1.n)$ $D_{10} = 2t\partial_t + x_a\partial_a + \dfrac{2}{\alpha_0}U\partial_U$ $Q_\alpha = \alpha_1 U\partial_U + \alpha_0 V\partial_V$
4	$\lambda_1 U_t = \Delta U$ $\lambda_2 V_t = \Delta V + \beta_2 U^{-\alpha_0}$	$\alpha_1\beta_2 \neq 0$ $\alpha_0 \neq 0; -1; -\dfrac{\lambda_2}{\lambda_1}$	$AE(1.n)$, D_{10}, X_2^∞ $Q_{\alpha_0} = -U\partial_U + \alpha_0 V\partial_V$,
5	$\lambda_1 U_t = \Delta U + \beta_1 U^{1-\alpha_0}$ $\lambda_2 V_t = \Delta V$	$\lambda_2\beta_1 \neq 0$ $\alpha_0 \neq 0; 1$	$AE(1.n)$, D_{10} $I = V\partial_V$, X_2^∞
6	$\lambda_1 U_t = \Delta U + \beta_1 U^{1-\alpha_0}$ $0 = \Delta V$	$\lambda_1\beta_1 \neq 0$ $\alpha_0 = 0; 1$	$AE(1.n)$, D_{10}, $I^\infty = T(t)V\partial_V$, $X_0^\infty = P_0(t, x)\partial_V$
7	$\lambda_1 U_t = \Delta U + \exp(-\gamma_0 U) f(\omega)$ $\lambda_2 V_t = \Delta V + V\exp(-\gamma_0 U) g(\omega)$	$\gamma_0 \neq 0$ $\omega = \dfrac{V}{\exp(\gamma U)}$	$AE(1.n)$, $D_2 = 2t\partial_t + x_a\partial_a$ $+\dfrac{2}{\gamma_0}(\partial_U + \gamma V\partial_V)$
8	$\lambda_1 U_t = \Delta U + \beta_1 \exp(-\gamma_0 U)$ $\lambda_2 V_t = \Delta V + \beta_2 \exp((\gamma - \gamma_0)U)$	$\gamma_0\beta_1\beta_2 \neq 0$	$AE(1.n)$, D_2 X_2^∞
9	$\lambda_1 U_t = \Delta U + \beta_1 V^{\alpha_1}\exp(-\gamma_0 U)$ $\lambda_2 V_t = \Delta V + \beta_2 V^{1+\alpha_1}\exp(-\gamma_0 U)$	$\alpha_1\gamma_0\beta_1 \neq 0$	$AE(1.n)$, $Q_\gamma = \alpha_1\partial_U + \gamma_0 V\partial_V$ $D_{20} = 2t\partial_t + x_a\partial_a + \dfrac{2}{\gamma_0}\partial_U$
10	$\lambda_1 U_t = \Delta U$ $\lambda_2 V_t = \Delta V + \beta_2 \exp(-\gamma_0 U)$	$\lambda_1 \neq 0$ $\gamma_0\beta_2 \neq 0$	$AE(1.n)$, D_{20}, X_2^∞ $Q_{\gamma_0} = -\partial_U + \gamma_0 V\partial_V$
11	$\lambda_1 U_t = \Delta U + \beta_1 \exp(-\gamma_0 U)$ $\lambda_2 V_t = \Delta V$	$\lambda_1\lambda_2 \neq 0$ $\gamma_0\beta_1 \neq 0$	$AE(1.n)$, D_{20} $I = V\partial_V$, X_2^∞
12	$\lambda_1 U_t = \Delta U + \beta_1 \exp(-\gamma_0 U)$ $0 = \Delta V$	$\lambda_1\beta_1\gamma_0 \neq 0$	$AE(1.n)$, D_{20}, $I^\infty = T(t)V\partial_V$ $X_0^\infty = P_0(t, x)\partial_V$
13	$\lambda_1 U_t = \Delta U + \exp(-\gamma_0 U) f(\omega)$ $\lambda_2 V_t = \Delta V + \exp(-\gamma_0 U) g(\omega)$	$\lambda_1\gamma_0 \neq 0$, $\omega = \gamma U - V$	$AE(1.n)$, $D_3 = 2t\partial_t + x_a\partial_a$ $+\dfrac{2}{\gamma_0}(\partial_U + \gamma\partial_V)$
14	$\lambda_1 U_t = \Delta U + \beta_1 \exp(\gamma_1 V - \gamma_0 U)$ $\lambda_2 V_t = \Delta V + \beta_2 \exp(\gamma_1 V - \gamma_0 U)$	$\gamma_0\gamma_1 \neq 0$ $\lambda_1\beta_2 \neq 0$	$AE(1.n)$, D_{20} $Q_\gamma^\infty = R_0(x)(\gamma_1\partial_U + \gamma_0\partial_V)$
15	$\lambda_1 U_t = \Delta U + \beta_1$ $\lambda_2 V_t = \Delta V + \beta_2 \exp(\gamma_1 V - \gamma_0 U)$	$\lambda_1\beta_2 \neq 0$ $\gamma_1\gamma_2 \neq 0$	$AE(1.n)$, Q_γ^∞ $D_{esp} = 2t\partial_t + x_a\partial_a - \dfrac{2}{\gamma_1}\partial_V$ $+\dfrac{2\beta_1}{\lambda_1 - \lambda_2}\left(t + \dfrac{\lambda_2\lvert x\rvert^2}{2n}\right)\left(\partial_U + \dfrac{\gamma_0}{\gamma_1}\partial_V\right)$
16	$\lambda_1 U_t = \Delta U + \beta_1 \exp(-\gamma_0 U)$ $\lambda_2 V_t = \Delta V + \beta_2 U$	$\gamma_0\beta_1\beta_2 \neq 0$	$AE(1.n)$, X_2^∞ $D_{esp} = 2t\partial_t + x_a\partial_a + \dfrac{2}{\gamma_0}\partial_U$ $+2\left(V - \dfrac{\beta_2\lvert x\rvert^2}{2\gamma_0 n}\right)\partial_V$

operators

$$
\begin{aligned}
&P_t = \partial_t \qquad P_x = \partial_x \qquad Q_\lambda = \lambda_1 U\partial_U + \lambda_2 V\partial_V \\
&G_x = tP_x - \tfrac{1}{2}xQ_\lambda \qquad D = 2tP_t + xP_x - \tfrac{1}{2}(U\partial_U + V\partial_V) \qquad (27)\\
&\Pi = t^2 P_t + tx_a P_a - \tfrac{1}{4}\lvert x\rvert^2 Q_\lambda - \tfrac{1}{2}t(U\partial_U + V\partial_V).
\end{aligned}
$$

Table 4. Other systems of the form (1).

	Systems	Restrictions	Basic operators of MAI
1	$\lambda_1 U_t = \Delta U + Uf(\omega)$ $\lambda_2 V_t = \Delta V + Vg(\omega)$	$\gamma \neq 0$; λ_2/λ_1 $\omega = U^{-\gamma}V$	$AE(1.n)$ $Q_\gamma = U\partial_U + \gamma V\partial_V$
2	$\lambda_1 U_t = \Delta U + Uf(\omega) + \beta_{10}U\log U$ $\lambda_2 V_t = \Delta V + Vg(\omega) + \beta_{20}V\log V$	$\lambda_1/\beta_{10} = \lambda_2/\beta_{20}$ $\lambda_2\beta_{20} \neq 0$	$AE(1.n)$ $\mathcal{Q}_\gamma = \exp\left(\frac{\beta_{10}}{\lambda_1}t\right)(U\partial_U + \gamma V\partial_V)$
3	$\lambda_1 U_t = \Delta U + Uf(\omega) + \beta_{10}U\log U$ $0 = \Delta V + Vg(\omega)$	$\lambda_1\beta_{10} \neq 0$	$AE(1.n)$, $\mathcal{Q}_\gamma$
4	$\lambda_1 U_t = \Delta U + \beta_1 U + \beta_{10}U\log U$ $0 = \Delta V + \beta_2 U^\gamma$	$\lambda_1\beta_2 \neq 0$ $\gamma \neq 0; 1$	$AE(1.n)$, $\mathcal{Q}_\gamma$ $X_0^\infty = P_0(t,x)\partial_V$
5	$\lambda_1 U_t = \Delta U + \beta_1 U + \beta_{10}U\log U$ $0 = \Delta V + \beta_2 V + \beta_{21}U$	$\lambda_1\beta_{10}\beta_{21} \neq 0$	$AE(1.n)$, $\mathcal{Q}_\gamma$, $\gamma = 1$ $X_{\beta_2}^\infty = P_{\beta_2}(t,x)\partial_V$
6	$\lambda_1 U_t = \Delta U + \beta_1 U + \beta_{10}U\log U$ $0 = \Delta V$	$\lambda_1\beta_{10} \neq 0$	$AE(1.n)$, $\mathcal{Q}_\gamma$, $\gamma = 1$ X_0^∞, $I^\infty = T(t)V\partial_V$
7	$\lambda_1 U_t = \Delta U + f(U)$ $\lambda_2 V_t = \Delta V + Vg(U) + \beta_{20}V\log V$	$\lambda_2\beta_{20} \neq 0$	$AE(1.n)$, $\mathcal{Q}^2 = \exp\left(\frac{\beta_{20}}{\lambda_2}t\right)V\partial_V$
8	$\lambda_1 U_t = \Delta U + \beta_1 U + \beta_{10}U\log U$ $\lambda_2 V_t = \Delta V + \beta_2 V + \beta_{20}V\log V$	$\lambda_1\beta_{10} \neq 0$ $\lambda_2\beta_{20} \neq 0$	$AE(1.n)$ $\mathcal{Q}^1 = \exp\left(\frac{\beta_{10}}{\lambda_1}t\right)U\partial_U$, $\mathcal{Q}^2$
9	$\lambda_1 U_t = \Delta U + \beta_1 U + \beta_{10}U\log U$ $\lambda_2 V_t = \Delta V$	$\lambda_1\lambda_2\beta_{10} \neq 0$	$AE(1.n)$, $\mathcal{Q}^1$, $I = V\partial_V$ $X_2^\infty = P_2(t,x)\partial_V$
10	$\lambda_1 U_t = \Delta U + f(U)$ $\lambda_2 V_t = \Delta V + Vg(U)$	$\lambda_2 \neq 0$	$AE(1.n)$, I
11	$\lambda_1 U_t = \Delta U + f(U)$ $\lambda_2 V_t = \Delta V$	$\lambda_2 \neq 0$	$AE(1.n)$, I, X_2^∞
12	$\lambda_1 U_t = \Delta U + f(U)$ $0 = \Delta V$	$\lambda_1 \neq 0$	$AE(1.n)$, $I^\infty = T(t)V\partial_V$ $X_0^\infty = P_0(t,x)\partial_V$,
13	$\lambda_1 U_t = \Delta U + f(U)$ $0 = \Delta V + Vg(U)$	$\lambda_1 \neq 0$	$AE(1.n)$, I^∞

According to the general procedure it is necessary to solve the Lagrange system

$$\frac{\mathrm{d}t}{\xi^0(t)} = \frac{\mathrm{d}x}{\xi^1(t,x)} = \frac{\mathrm{d}U}{\eta_1(t,x)U} = \frac{\mathrm{d}V}{\eta_2(t,x)V} \tag{28}$$

where $\xi^0, \xi^1, \eta_1 U, \eta_2 V$ are known coefficients of the infinitesimal operator X which are obtained as a linear combination of the operators (27).

It is known, [10, 11], that a full set of non-equivalent (non-conjugate) one-dimensional subalgebras of the $AG_2(1.1)$ algebra is generated by the operators

$$\begin{aligned} &X_1 = Q_\lambda \qquad X_2 = P_x \qquad X_3 = P_t - \alpha Q_\lambda \\ &X_4 = P_t + \delta G_x \qquad X_5 = D - \alpha Q_\lambda \qquad X_6 = P_t + \Pi - \alpha Q_\lambda \end{aligned} \tag{29}$$

where $\alpha \in \mathbb{R}$, $\delta = \pm 1$. It should be noted that this is not a unique representation of such a set.

Table 5. Other systems of the form (1).

	Systems	Restrictions	Basic operators of MAI
1	$\lambda_1 U_t = \Delta U + f(\omega)$ $\lambda_2 V_t = \Delta V + V g(\omega)$	$\lambda_2\gamma \neq 0$ $\omega = V/\exp\gamma U$	$AE(1.n)$, $Q_\gamma = \partial_U + \gamma V \partial_V$
2	$\lambda_1 U_t = \Delta U + f(\omega)$ $0 = \Delta V + V g(\omega)$	$\lambda_1\gamma \neq 0$	$AE(1.n)$ $Q_\gamma^\infty = T(t)(\partial_U + \gamma V \partial_V)$
3	$\lambda_1 U_t = \Delta U + \beta_1 U + f(\omega)$ $\lambda_2 V_t = \Delta V + \beta_{20} V \log V + V g(\omega)$	$\lambda_2\beta_{20}\gamma \neq 0$ $\beta_1/\lambda_1 = \beta_{20}/\lambda_2$	$AE(1.n)$ $\mathcal{Q}_\gamma = \exp\left(\frac{\beta_1}{\lambda_1}t\right)(\partial_U + \gamma V \partial_V)$
4	$\lambda_1 U_t = \Delta U + \beta_1 U + f(\omega)$ $0 = \Delta V + V g(\omega)$	$\lambda_1\beta_1\gamma \neq 0$	$AE(1.n)$ $\mathcal{Q}_\gamma = \exp\left(\frac{\beta_1}{\lambda_1}t\right)(\partial_U + \gamma V \partial_V)$
5	$0 = \Delta U + f(\omega)$ $\lambda_2 V_t = \Delta V + \beta_{20} V \log V + V g(\omega)$	$\lambda_2\beta_{20} \neq 0$ $\omega = V/\exp\gamma U$	$AE(1.n)$ $\mathcal{Q}_\gamma = \exp\left(\frac{\beta_{20}}{\lambda_2}t\right)(\partial_U + \gamma V \partial_V)$
6	$\lambda_1 U_t = \Delta U + \beta_1 U + \beta_{10}\log V$ $\lambda_2 V_t = \Delta V + \beta_2 V + \beta_{20} V \log V$	$\lambda_1\beta_{20} \neq \lambda_2\beta_1$ $\lambda_2\beta_{10}\beta_{20} \neq 0$	$AE(1.n)$, $X_{\beta_1}^\infty = P_{\beta_1}(t,x)\partial_U$ $\mathcal{Q}_\gamma = \exp\left(\frac{\beta_{20}}{\lambda_2}t\right)\left(\partial_U + \frac{\lambda_1\beta_{20} - \lambda_2\beta_1}{\lambda_2\beta_{10}} V \partial_V\right)$
7	$\lambda_1 U_t = \Delta U + \beta_1 U + \beta_{10}\log V$ $\lambda_2 V_t = \Delta V + \beta_2 V + \beta_{20} V \log V$	$\lambda_1\beta_{20} = \lambda_2\beta_1$ $\lambda_2\beta_{10}\beta_{20} \neq 0$	$AE(1.n)$, $X_{\beta_1}^\infty$ $\mathcal{Q}_\gamma^t = \exp\left(\frac{\beta_{20}}{\lambda_2}t\right)\left(t\partial_U + \frac{\lambda_1}{\beta_{10}} V \partial_V\right)$
8	$\lambda_1 U_t = \Delta U + \beta_1 U + f(V)$ $\lambda_2 V_t = \Delta V + g(V)$		$AE(1.n)$ $X_{\beta_1}^\infty$
9	$\lambda_1 U_t = \Delta U + \beta_1 U + f(\alpha U - V)$ $\lambda_2 U_t = \Delta V + \beta_2 V + g(\alpha U - V)$	$\alpha \neq 0$	$AE(1.n)$, $Z^\infty = \exp\left(\frac{\beta_1 - \beta_2}{\lambda_1 - \lambda_2}t\right)$ $\times R(x)(\partial_U + \alpha\partial_V)$

Solving the system (28) for the operators $X_1, \ldots, X_6$, respectively, we obtain a set of non-equivalent Lie ansätze for the functions U and V:

$$
\begin{aligned}
&X_1: \quad V^{\lambda_1} = \rho U^{\lambda_2} \qquad \rho \in \mathbb{R}\\
&X_2: \quad U = \varphi_1(t) \qquad V = \varphi_2(t)\\
&X_3: \quad U = \varphi_1(x)\exp(-\alpha\lambda_1 t) \qquad V = \varphi_2(x)\exp(-\alpha\lambda_2 t)\\
&X_4: \quad U = \exp\left[\tfrac{1}{2}\lambda_1 t\left(\tfrac{1}{3}t^2 - \delta x\right)\right]\varphi_1(\omega)\\
&\qquad\;\; V = \exp\left[\tfrac{1}{2}\lambda_2 t\left(\tfrac{1}{3}t^2 - \delta x\right)\right]\varphi_2(\omega) \qquad \omega = 2x - \delta t^2\\
&X_5: \quad U = t^{-(2\alpha\lambda_1+1)/4}\varphi_1(\omega) \qquad V = t^{-(2\alpha\lambda_2+1)/4}\varphi_2(\omega) \qquad \omega = x/\sqrt{t}\\
&X_6: \quad U = (t^2+1)^{-1/4}\exp\left[-\frac{\lambda_1}{4}\left(\frac{tx^2}{1+t^2} + 4\alpha\arctan t\right)\right]\varphi_1(\omega)\\
&\qquad\;\; V = (t^2+1)^{-1/4}\exp\left[-\frac{\lambda_2}{4}\left(\frac{tx^2}{1+t^2} + 4\alpha\arctan t\right)\right]\varphi_2(\omega) \qquad \omega = \frac{x}{\sqrt{1+t^2}}.
\end{aligned}
\tag{30}
$$

Using the ansätze (30), we can reduce the nonlinear system (7) to systems of ordinary differential equations (ODEs). In contrast to the other ansätze, the one for the operator X_1 leads, however, to a system of partial differential equations

$$
\begin{aligned}
\lambda_1 U_t &= U_{xx} + \beta_1\rho^{\gamma_1} U\\
\lambda_2 V_t &= V_{xx} + \beta_2\rho^{\gamma_1} V.
\end{aligned}
\tag{31}
$$

The reduced systems of ODEs for the other five ansätze are as follows:

$$\lambda_1 \frac{d\varphi_1}{dt} = \beta_1 \varphi_1^{1-\gamma_1\lambda_2} \varphi_2^{\gamma_1\lambda_1} \qquad \lambda_2 \frac{d\varphi_2}{dt} = \beta_2 \varphi_2^{1+\gamma_1\lambda_1} \varphi_1^{-\gamma_1\lambda_2} \tag{32}$$

$$\frac{d^2\varphi_1}{dx^2} + \alpha\lambda_1^2\varphi_1 + \beta_1\varphi_1^{1-\gamma_1\lambda_2}\varphi_2^{\gamma_1\lambda_1} = 0 \qquad \frac{d^2\varphi_2}{dx^2} + \alpha\lambda_2^2\varphi_2 + \beta_2\varphi_2^{1+\gamma_1\lambda_1}\varphi_1^{-\gamma_1\lambda_2} = 0 \tag{33}$$

$$4\frac{d^2\varphi_1}{d\omega^2} + \frac{1}{4}\delta\lambda_1^2\omega\varphi_1 + \beta_1\varphi_1^{1-\gamma_1\lambda_2}\varphi_2^{\gamma_1\lambda_1} = 0 \qquad 4\frac{d^2\varphi_2}{d\omega^2} + \frac{1}{4}\delta\lambda_2^2\omega\varphi_2 + \beta_2\varphi_2^{1+\gamma_1\lambda_1}\varphi_1^{-\gamma_1\lambda_2} = 0 \tag{34}$$

$$\frac{d^2\varphi_1}{d\omega^2} + \frac{1}{2}\lambda_1\omega\frac{d\varphi_1}{d\omega} + \frac{\lambda_1}{4}(2\alpha\lambda_1 + 1)\varphi_1 + \beta_1\varphi_1^{1-\gamma_1\lambda_2}\varphi_2^{\gamma_1\lambda_1} = 0 \qquad \frac{d^2\varphi_2}{d\omega^2} + \frac{1}{2}\lambda_2\omega\frac{d\varphi_2}{d\omega} + \frac{\lambda_2}{4}(2\alpha\lambda_2 + 1)\varphi_2 + \beta_2\varphi_2^{1+\gamma_1\lambda_1}\varphi_1^{-\gamma_1\lambda_2} = 0 \tag{35}$$

$$\frac{d^2\varphi_1}{d\omega^2} + \frac{\lambda_1^2}{4}(4\alpha + \omega^2)\varphi_1 + \beta_1\varphi_1^{1-\gamma_1\lambda_2}\varphi_2^{\gamma_1\lambda_1} = 0 \qquad \frac{d^2\varphi_2}{d\omega^2} + \frac{\lambda_2^2}{4}(4\alpha + \omega^2)\varphi_2 + \beta_2\varphi_2^{1+\gamma_1\lambda_1}\varphi_1^{-\gamma_1\lambda_2} = 0 \tag{36}$$

where φ_1, φ_2 are new unknown functions on one variable and $\gamma_1 = 4/(\lambda_1 - \lambda_2)$.

Having exact solutions of these systems of ODEs and using the relevant ansätze from (30), one obtains solutions of the nonlinear reaction–diffusion system (7). Note that the system (31) obtained from (7) by the operator X_1 reduction, is *a linear system* coupled by the functional condition $V^{\lambda_1} = \rho U^{\lambda_2}$. This system can be reduced to an overdetermined one, made up of a linear diffusion equation and a Hamilton–Jacobi-type equation of the form

$$V^{\lambda_1} = \rho U^{\lambda_2} \qquad \lambda_1 U_t = U_{xx} + \beta_1\rho^{\gamma_1}U \qquad \lambda_1 U_t = \frac{U_x^2}{U} + \frac{\lambda_1\gamma_1}{4\lambda_2}(\beta_1\lambda_2 - \beta_2\lambda_1)\rho^{\gamma_1}U. \tag{37}$$

It turns out that this system is integrated only in the case $\kappa \equiv \beta_1\lambda_2^2 - \beta_2\lambda_1^2 = 0$ and the relevant general solution has the following form:

$$U = \exp\left(\lambda_1\left(c_0 + c_1 x + c_1^2 t + \frac{\beta_1}{\lambda_1^2}\rho^{\gamma_1}t\right)\right) \qquad V = \rho^{1/\lambda_1}\exp\left(\lambda_2\left(c_0 + c_1 x + c_1^2 t + \frac{\beta_2}{\lambda_2^2}\rho^{\gamma_1}t\right)\right) \tag{38}$$

where ρ, c_0 and c_1 are arbitrary constants.

The ODE system (32) can also be integrated (see below). Regarding the systems (33)–(36), we can only say that they are not integrable. However, a successful way to find particular solutions is to use the substitution $\varphi_1 = \rho(\omega)\exp(\lambda_1 W(\omega))$, $\varphi_2 = \rho(\omega)\exp(\lambda_2 W(\omega))$. It turns out that this substitution reduces every system (33)–(36) to one with a similar structure to the equations obtained in [19], where a procedure for finding particular solutions of such ODEs was suggested.

Since (29) is a full set of non-equivalent (non-conjugate) one-dimensional subalgebras of $AG_2(1.1)$, any invariant solution of the nonlinear system (7) can be obtained using one of the ansätze (30), where φ_1, φ_2 are the relevant solutions of (31)–(36). To achieve this it suffices to apply additionally continuous transformations generated by the basic operators of the MAI (27). The general form of such transformations can be found in the following way. Let us consider an arbitrary solution $(U^0(t,x), V^0(t,x))$ of the system (7). Then a successive application of the above-mentioned transformations for this solution leads to a six-parameter family of solutions (similar formulae for nonlinear Schrödinger equations were found in [16, 19]):

$$
\begin{aligned}
U_{new} &= \exp\left[\lambda_1 \frac{pm^2x^2 + 2m\varepsilon^1 x + m^2\varepsilon^2 t + b_0}{4(d_0 - pm^2 t)}\right] \\
&\quad \times \frac{m_0^{\lambda_1} m^{1/2}}{(d_0 - pm^2t)^{1/2}} U^0\left(\frac{m^2t + d_0^1}{d_0 - pm^2t}, \frac{mx + m^2\varepsilon t + d}{d_0 - pm^2t}\right) \\
V_{new} &= \exp\left[\lambda_2 \frac{pm^2x^2 + 2m\varepsilon^1 x + m^2\varepsilon^2 t + b_0}{4(d_0 - pm^2 t)}\right] \\
&\quad \times \frac{m_0^{\lambda_2} m^{1/2}}{(d_0 - pm^2t)^{1/2}} V^0\left(\frac{m^2t + d_0^1}{d_0 - pm^2t}, \frac{mx + m^2\varepsilon t + d}{d_0 - pm^2t}\right)
\end{aligned}
\tag{39}
$$

where $d_0 = 1 - pd_0^1$, $d = d^1 + \varepsilon d_0^1$, $\varepsilon^1 = \varepsilon + pd^1$, $b_0 = p(d^1)^2 + 2\varepsilon d^1 + \varepsilon^2 d_0^1$ and ε, p, $m_0 > 0$, $m > 0$, d_0^1, d^1 are arbitrary parameters.

Interesting particular cases of the formula (39) are

$$
\begin{aligned}
U_{new} &= U^0(t, x + \varepsilon t) \exp\left(\tfrac{1}{2}\lambda_1\left(\varepsilon x + \tfrac{1}{2}\varepsilon^2 t\right)\right) \\
V_{new} &= V^0(t, x + \varepsilon t) \exp\left(\tfrac{1}{2}\lambda_2\left(\varepsilon x + \tfrac{1}{2}\varepsilon^2 t\right)\right)
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
U_{new} &= t^{-1/2} \exp\left(\frac{-\lambda_1 x^2}{4t}\right) U^0\left(-\frac{1}{t}, \frac{x}{t}\right) \\
V_{new} &= t^{-1/2} \exp\left(\frac{-\lambda_2 x^2}{4t}\right) V^0\left(-\frac{1}{t}, \frac{x}{t}\right).
\end{aligned}
\tag{41}
$$

Formula (40) is generated by the Galilei transformation and the ε parameter can be thought of as representing a velocity. Formula (41) can be obtained by the passage to the limit $p \to \infty$, $m \to 0$, $pm \to -1$ (the other parameters being zero). Both formulae can be used to convert time-independent (stationary) solutions of the nonlinear system (7) into time-dependent (non-stationary) ones.

Remark 4. The formulae for the multiplication of solutions that are presented above for the $(1+1)$-dimensional case can easily be generalized to the multidimensional case in which one considers the system (24). In particular, formula (41) has the multidimensional analogue

$$
\begin{aligned}
U_{new} &= t^{-n/2} \exp\left(\frac{-\lambda_1 |x|^2}{4t}\right) U^0\left(-\frac{1}{t}, \frac{x}{t}\right) \\
V_{new} &= t^{-n/2} \exp\left(\frac{-\lambda_2 |x|^2}{4t}\right) V^0\left(-\frac{1}{t}, \frac{x}{t}\right)
\end{aligned}
\tag{42}
$$

where $|x|^2 = x_1^2 + \cdots + x_n^2$.

Formula (41) contains no parameters, so it is not an expression for the multiplication of an exact solution into a family of solutions but a formula for the transfer of one solution into another. It should be noted that a similar result for the linear diffusion equation was obtained by Appell [20] and its analogues for nonlinear Schrödinger equations were constructed in [16, 19].

Finally, let us present an example of the application of (41). The system of ODEs (32) is easily integrated and its general solution leads to the following solutions of (7):

$$U^0 = A^{\lambda_1}\left(\pm\frac{\lambda_1\lambda_2}{\gamma_1\kappa}\right)^{1/4}(t_0 \pm t)^{\beta_1\lambda_2/\gamma_1\kappa}$$
$$V^0 = A^{\lambda_2}\left(\pm\frac{\lambda_1\lambda_2}{\gamma_1\kappa}\right)^{1/4}(t_0 \pm t)^{\beta_2\lambda_1/\gamma_1\kappa} \qquad (43)$$

if $\kappa = \beta_1\lambda_2^2 - \beta_2\lambda_1^2 \neq 0$, and

$$U^0 = A^{\lambda_1}\left(\frac{\lambda_1^2\alpha}{\beta_1}\right)^{1/4}\exp(\lambda_1\alpha t)$$
$$V^0 = A^{\lambda_2}\left(\frac{\lambda_1^2\alpha}{\beta_1}\right)^{1/4}\exp(\lambda_2\alpha t) \qquad (44)$$

if $\kappa = 0$; the constants t_0, A and α are arbitrary. Using formula (41), solutions (43) and (44) are converted into solutions of the form:

$$U_{new} = A^{\lambda_1}\left(\pm\frac{\lambda_1\lambda_2}{\gamma_1\kappa}\right)^{1/4}t^{-1/2}\left(t_0 \mp \frac{1}{t}\right)^{\beta_1\lambda_2/\gamma_1\kappa}\exp\left(-\frac{\lambda_1 x^2}{4t}\right)$$
$$V_{new} = A^{\lambda_2}\left(\pm\frac{\lambda_1\lambda_2}{\gamma_1\kappa}\right)^{1/4}t^{-1/2}\left(t_0 \mp \frac{1}{t}\right)^{\beta_2\lambda_1/\gamma_1\kappa}\exp\left(-\frac{\lambda_2 x^2}{4t}\right) \qquad (45)$$

and

$$U_{new} = A^{\lambda_1}\left(\frac{\lambda_1^2\alpha}{\beta_1}\right)^{1/4}t^{-1/2}\exp\left(-\frac{\lambda_1}{t}\left(\alpha + \tfrac{1}{4}x^2\right)\right)$$
$$V_{new} = A^{\lambda_2}\left(\frac{\lambda_1^2\alpha}{\beta_1}\right)^{1/4}t^{-1/2}\exp\left(-\frac{\lambda_2}{t}\left(\alpha + \tfrac{1}{4}x^2\right)\right). \qquad (46)$$

It is worth commenting on some of the properties of (46) and (45). For $\beta_1 > 0$, we require $\alpha > 0$ in (46) and the solution is identically zero at $t = 0$ (providing an example of a non-uniqueness property for the corresponding system of reaction–diffusion equations) and decays as $t \to +\infty$ in the form of two independent Gaussians (corresponding to linear diffusive behaviour). For $\beta_1 < 0$, we require $\alpha < 0$, leading to solutions which blow up in the range $x^2 < -4\alpha$, but decay to zero elsewhere, as $t \to 0^+$. In (45), the lower signs are needed for $\gamma_1\kappa < 0$ and one has identically zero initial data (again implying non-uniqueness) if $\beta_1\lambda_2, \beta_2\lambda_1 > -\gamma_1\kappa/2$. For $t_0 < 0$, U blows up as $t \to -1/t_0$ if $\beta_1 > 0$ and extinguishes if $\beta_1 < 0$; similarly V blows up if $\beta_2 > 0$ and extinguishes if $\beta_2 < 0$. For $t_0 > 0$, both U and V decay in a linear (Gaussian) fashion as $t \to +\infty$; in the (non-generic) borderline case $t_0 = 0$ the solutions remain bounded but the mass of U tends to infinity as $t \to +\infty$ if $\beta_1 > 0$ and to zero if $\beta_1 < 0$, with that of V having the corresponding dependence on the sign of β_2. This family of solutions is thus particularly interesting in that it provides explicit illustrations of each of these asymptotic outcomes; in view of its very special symmetry properties it seems likely, however, that (7) has a non-generic status (within the class of systems (1) with power-law nonlinearities) with respect to its asymptotic behaviour. For $\gamma_1\kappa > 0$ (which requires $\lambda_2 < \lambda_1 < \sqrt{\beta_1/\beta_2}\,\lambda_2$ or $\lambda_1 < \lambda_2 < \sqrt{\beta_2/\beta_1}\,\lambda_1$), the upper signs are needed in (45) and Gaussian behaviour ensues as $t \to +\infty$.

The solutions (45) and (46) have a similar character to the *fundamental solution* of the linear diffusion equation ((25) with $n = 1$). We note that in the case of a single nonlinear reaction–diffusion equation

$$U_t = U_{xx} + C(U) \tag{47}$$

there are no known exact solutions with similar structure.

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